

SMOOTH DENSITIES OF THE LAWS OF PERTURBED DIFFUSION PROCESSES

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ABSTRACT. Under some regularity conditions on b, σ and α , we prove that the following perturbed stochastic differential equation

$$(0.1) \quad X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s + \alpha \sup_{0 \leq s \leq t} X_s, \quad \alpha < 1$$

admits smooth densities for all $0 \leq t \leq t_0$, where $t_0 > 0$ is some finite number.

Keywords: Perturbed diffusion processes, Malliavin differentiability, Smooth density.

Mathematics Subject Classification (2000): 60H07.

1. INTRODUCTION

There have been a considerable body of literatures devoted to the study of perturbed stochastic differential equations(SDEs), see [1]-[7],[9],[11]. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$, let $\{B_t\}_{t \geq 0}$ be a one-dimensional standard $\{\mathcal{F}_t\}_{t \geq 0}$ -Brownian Motion. Suppose that $\sigma(x), b(x)$ are Lipschitz continuous functions on \mathbb{R} . It was proved in [5] that the following perturbed stochastic differential equation:

$$(1.1) \quad X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s + \alpha \sup_{0 \leq s \leq t} X_s, \quad \forall \alpha < 1,$$

admits a unique solution. If $|\sigma(x)| > 0$, it was shown in [11] that the law of X_t is absolutely continuous with respect to Lebesgue measure, i.e. the law of X_t admits a density for $t > 0$.

There seem no smooth density results for the law of a perturbed diffusion process, this paper aims to partly fill in this gap. The smoothness of densities is a popular topic in stochastic analysis and has been intensively studied for several decades, we refer readers to [8], [10] and references therein. Our approach to proving the smoothness of densities is by Malliavin calculus, so let us first recall some well known results on Malliavin calculus [8] to be used in this paper.

Let $\Omega = C_0(\mathbb{R}_+)$ be the space of continuous functions on \mathbb{R}_+ which are zero at zero. Denote by \mathcal{F} the Borel σ -field on Ω and \mathbb{P} the Wiener measure, then the canonical

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coordinate process $\{\omega_t, t \in \mathbb{R}_+\}$ on Ω is a Brownian motion B_t . Define $\mathcal{F}_t^0 = \sigma(B_s, s \leq t)$ and denote by \mathcal{F}_t the completion of \mathcal{F}_t^0 with respect to the \mathbb{P} -null sets of \mathcal{F} .

Let $H := L^2(\mathbb{R}_+, \mathcal{B}, \mu)$ where $(\mathbb{R}_+, \mathcal{B})$ is a measurable space with \mathcal{B} being the Borel σ -field of \mathbb{R}_+ and μ being the Lebesgue measure on \mathbb{R}_+ , we denote the norm of H by $\|\cdot\|_H$. For any $h \in H$, $W(h)$ is defined by

$$(1.2) \quad W(h) = \int_0^\infty h(t) dB_t,$$

note that $\{W(h), h \in H\}$ is a Gaussian Process on H .

We denote by $C_p^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that f and all of its partial derivatives have polynomial growth. Let \mathcal{S} be the set of smooth random variables defined by

$$\mathcal{S} = \{F = f(W(h_1), \dots, W(h_n)); h_1, \dots, h_n \in H, n \geq 1, f \in C_p^\infty(\mathbb{R}^n)\}.$$

Let $F \in \mathcal{S}$, define its Malliavin derivative $D_t F$ by

$$(1.3) \quad D_t F = \sum_{i=1}^n \partial_i f(W(h_1), \dots, W(h_n)) h_i(t),$$

and its norm by

$$\|F\|_{1,2} = [\mathbb{E}(|F|^2) + \mathbb{E}(\|DF\|_H^2)]^{\frac{1}{2}},$$

where $\|DF\|_H^2 = \int_0^\infty |D_t F|^2 \mu(dt)$. Denote by $\mathbb{D}^{1,2}$ the completion of \mathcal{S} under the norm $\|\cdot\|_{1,2}$. We further define the norm

$$\|F\|_{m,2} = \left[\mathbb{E}(|F|^2) + \sum_{k=1}^m \mathbb{E}(\|D^k F\|_{H^{\otimes k}}^2) \right]^{\frac{1}{2}}.$$

Similarly, $\mathbb{D}^{m,2}$ denotes the completion of \mathcal{S} under the norm $\|\cdot\|_{m,2}$.

We shall use the following two propositions:

Proposition 1.1 (Proposition 1.2.3 of [8]). *Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Suppose that $F = (F^1, \dots, F^d)$ is a random vector whose components belong to the space $\mathbb{D}^{1,2}$. Then $\phi(F) \in \mathbb{D}^{1,2}$, and*

$$D(\phi(F)) = \sum_{i=1}^d \partial_i \phi(F) D F^i.$$

Proposition 1.2 (Proposition 2.1.5 of [8]). *If $F \in \mathbb{D}^{\infty,2}$ with $\mathbb{D}^{\infty,2} = \cap_{m \geq 1} \mathbb{D}^{m,2}$ and $\|DF\|_H^{-1} \in \cap_{p \geq 1} L^p(\Omega)$, then the density of F belongs to the infinitely continuously differentiable function space $C^\infty(\mathbb{R})$.*

Throughout this paper, for a bounded measurable function f , we shall denote

$$\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|.$$

2. MAIN RESULTS

Throughout this paper, we need to assume $\alpha < 1$ to guarantee that Eq. (1.1) has a unique solution [5]. Furthermore, it is shown in [11] that

Theorem 2.1. ([11, Theorem 3.1]) *Let $(X_t)_{t \geq 0}$ be the unique solution to Eq. (1.1). Then $X_t \in \mathbb{D}^{1,2}$ for all $t > 0$.*

Theorem 2.2. ([11, Theorem 3.2]) *Assume that σ and b are both Lipschitz continuous, and $|\sigma(x)| > 0$ for all $x \in \mathbb{R}$. Then, for $t > 0$, the law of X_t is absolutely continuous with respect to Lebesgue measure.*

In this paper, we shall prove the following results about the smoothness of densities:

Theorem 2.3. *Assume that b is bounded smooth and that $\sigma(x) \equiv \sigma$. If $\alpha < 1$, $t_0 > 0$ and b satisfy*

$$\theta(t_0, \alpha, b) < 1/2,$$

with $\theta(t_0, \alpha, b) := \sqrt{2\|b'\|_\infty^2 t_0^2 + 8\alpha^2} + \|b'\|_\infty^2 t_0^2 + 4\alpha^2$, then the law of X_t in (1.1) admits a smooth density for all $t \in (0, t_0]$.

Theorem 2.4. *Assume that b is bounded smooth, and σ is bounded smooth with $\|\sigma'\|_\infty < \infty$, $\|\sigma''\|_\infty < \infty$ and $\inf_{x \in \mathbb{R}} |\sigma(x)| > 0$. Let*

$$(2.1) \quad F(y) = \int_x^y \frac{1}{\sigma(u)} du, \quad y \in (-\infty, \infty)$$

and $\tilde{b}(x) = \frac{b(F^{-1}(x))}{\sigma(F^{-1}(x))} - \frac{1}{2}\sigma'(F^{-1}(x))$, then \tilde{b} is bounded smooth with $\|\tilde{b}\|_\infty < \infty$. If $\alpha < 1$, $t_0 > 0$ and b satisfy

$$\theta(t_0, \alpha, \tilde{b}) < 1/2$$

with $\theta(t_0, \alpha, \tilde{b}) := \sqrt{2\|\tilde{b}'\|_\infty^2 t_0^2 + 8\alpha^2} + \|\tilde{b}'\|_\infty^2 t_0^2 + 4\alpha^2$, then the law of X_t in (1.1) admits a smooth density for all $t \in (0, t_0]$.

Proofs of Theorems 2.3 and 2.4: The main idea is to use Proposition 1.2 to prove the two theorems. To verify the conditions in Proposition 1.2, it suffices to prove that $X_t \in D^{m,2}$ for all $m \geq 1$ and $\|DX_t\|_H \geq c > 0$ a.s. for some constant $c > 0$.

Theorem 2.3 immediately follows from Lemmas 3.1 and 3.4 below.

Now we prove Theorem 2.4. Recall $Y_t = \int_x^{X_t} \frac{1}{\sigma(u)} du$ in Lemma 3.5 below, by the condition of σ , F is a continuous and strictly increasing function with bounded derivative and thus

$$(2.2) \quad \|DY_t\|_H = \|DF(X_t)\|_H \leq \frac{1}{\inf_{x \in \mathbb{R}} |\sigma(x)|} \|DX_t\|_H.$$

Hence, by Lemmas 3.1 and 3.5 below, under the same condition as in Theorem 2.4 we have

$$(2.3) \quad \|DX_t\|_H \geq \inf_{x \in \mathbb{R}} |\sigma(x)| \cdot \|DY_t\|_H \geq \inf_{x \in \mathbb{R}} |\sigma(x)| \cdot \frac{[1 - 2\theta(t_0, \alpha, \tilde{b})]t}{2(1 + 2\|\tilde{b}'\|_\infty^2 t^2 + 2\alpha^2)} \quad t \in [0, t_0].$$

Hence, X_t admits a smooth density for all $t \in (0, t_0]$. □

3. AUXILIARY LEMMAS

It is well known that $\|DX_t\|_H$ has the following representation [11] for all $t > 0$:

$$\|DX_t\|_H = \left(\int_0^t |D_r X_t|^2 dr \right)^{\frac{1}{2}}$$

with $D_r X_t$ satisfying

$$(3.1) \quad D_r X_t = \sigma(X_r) + \int_r^t D_r b(X_s) ds + \int_r^t D_r \sigma(X_s) dB_s + \alpha D_r \left(\sup_{0 \leq s \leq t} X_s \right).$$

We shall often use the following fact ([11], [8])

$$(3.2) \quad D_r X_t = 0 \quad \text{if } r > t,$$

$$(3.3) \quad \left\| D \left(\sup_{0 \leq s \leq t} X_s \right) \right\|_H \leq \sup_{0 \leq s \leq t} \|DX_s\|_H,$$

where

$$\left\| D \left(\sup_{0 \leq s \leq t} X_s \right) \right\|_H^2 = \int_0^t \left| D_r \left(\sup_{0 \leq s \leq t} X_s \right) \right|^2 dr, \quad \|DX_t\|_H^2 = \int_0^t |D_r X_t|^2 dr.$$

3.1. X_t is an element in $\mathbb{D}^{m,2}$ for all $t > 0$ and $m \geq 1$.

Lemma 3.1. *Let X_t be the solution of the perturbed stochastic differential equation (1.1), and suppose that the coefficients b and σ are smooth with bounded derivatives of all orders. Then X_t belongs to $\mathbb{D}^{m,2}$ for all $t > 0$ and all $m \geq 1$.*

Proof. We shall use Picard iteration to prove the lemma. Letting $X_t^0 = x_0$ for all $t > 0$, define X_t^{n+1} be the unique, adapted solution to the following equation:

$$(3.4) \quad X_t^{n+1} = x_0 + \int_0^t \sigma(X_s^n) dB_s + \int_0^t b(X_s^n) ds + \alpha \max_{0 \leq s \leq t} (X_s^{n+1}),$$

which obviously implies

$$\max_{0 \leq s \leq t} (X_s^{n+1}) = x_0 + \max_{0 \leq s \leq t} \left(\int_0^t \sigma(X_s^n) dB_s + \int_0^t b(X_s^n) ds \right) + \alpha \max_{0 \leq s \leq t} (X_s^{n+1}).$$

Therefore,

$$\max_{0 \leq s \leq t} (X_s^{n+1}) = \frac{x_0}{1 - \alpha} + \frac{1}{1 - \alpha} \max_{0 \leq s \leq t} \left(\int_0^t \sigma(X_s^n) dB_s + \int_0^t b(X_s^n) ds \right),$$

this and (3.4) further gives

$$X_t^{n+1} = \frac{x_0}{1-\alpha} + \int_0^t \sigma(X_s^n) dB_s + \int_0^t b(X_s^n) ds \\ + \frac{\alpha}{1-\alpha} \max_{0 \leq s \leq t} \left(\int_0^s \sigma(X_u^n) dB_u + \int_0^s b(X_u^n) du \right).$$

By the above representation of X_t^{n+1} and a standard method [5], for every $t > 0$ we have

$$(3.5) \quad \lim_{n \rightarrow \infty} X_t^n = X_t \quad \text{in } L^2(\Omega).$$

Let $m \geq 1$, it is standard to check that $X_t^n \in \mathbb{D}^{m,2}$ for every $t > 0$ and $n \geq 1$ [11, Theorem 3.1]. By a similar argument as in [11, Theorem 3.1], we have

$$(3.6) \quad \sup_{n \geq 1} \mathbb{E} [\|D^k X_t^n\|_{H^{\otimes k}}^2] < \infty, \quad k = 1, \dots, m.$$

Next we prove $X_t \in \mathbb{D}^{m,2}$ by the argument of [8, Proposition 1.2.3]. Indeed, by (3.6), there exists some subsequence $D^k X_t^{n_j}$ weakly converges to some α_k in $L^2(\Omega, H^{\otimes k})$ for $k = 1, \dots, m$. By (3.5) and the remark immediately below [8, Proposition 1.2.2], the projections of $D^k X_t^{n_j}$ on any Wiener chaos converge in the weak topology of $L^2(\Omega)$, as n_j tends to infinity, to those of α_k for $k = 1, \dots, m$. Hence, $X_t \in \mathbb{D}^{m,2}$ and $D^k X_t = \alpha_k$ for $k = 1, \dots, m$. Moreover, for any weakly convergent subsequence the limit must be equal to $\alpha_1, \dots, \alpha_m$ by the same argument as above, and this implies the weak convergence of the whole sequence. \square

3.2. Additive noise case. If $\sigma(x) \equiv \sigma$, then Eq. (3.1) reads as

$$(3.7) \quad D_r X_t = \sigma + \int_r^t D_r b(X_s) ds + \alpha D_r \left(\sup_{0 \leq s \leq t} X_s \right).$$

Lemma 3.2. *Let $t > 0$ be arbitrary and b be bounded smooth with $\|b'\|_\infty < \infty$. For all $0 < t_1 < t_2 \leq t$, we have*

$$|\|DX_{t_2}\|_H^2 - \|DX_{t_1}\|_H^2| \leq 2 \left[\sqrt{2\|b'\|_\infty^2(t_2 - t_1)^2 + 8\alpha^2} + \|b'\|_\infty^2(t_2 - t_1)^2 + 4\alpha^2 \right] \sup_{0 \leq s \leq t} \|DX_s\|_H^2.$$

Proof. It is easy to see that

$$|\|DX_{t_2}\|_H^2 - \|DX_{t_1}\|_H^2| = \left| \int_0^{t_2} (D_r X_{t_2})^2 dr - \int_0^{t_1} (D_r X_{t_1})^2 dr \right| \leq I_1 + I_2,$$

where

$$I_1 := \int_{t_1}^{t_2} (D_r X_{t_2})^2 dr, \quad I_2 := \int_0^{t_1} |(D_r X_{t_2})^2 - (D_r X_{t_1})^2| dr.$$

We claim that

$$(3.8) \quad \int_0^{t_2} (D_r X_{t_2} - D_r X_{t_1})^2 dr \leq 2 [\|b'\|_\infty^2 (t_2 - t_1)^2 + 4\alpha^2] \sup_{0 \leq s \leq t} \|DX_s\|_H^2.$$

and we will prove it in the last part of this proof.

Let us now estimate I_1 and I_2 by (3.8). Observe

$$I_1 = \int_{t_1}^{t_2} (D_r X_{t_2} - D_r X_{t_1})^2 dr \leq \int_0^{t_2} (D_r X_{t_2} - D_r X_{t_1})^2 dr,$$

by (3.8) we have

$$(3.9) \quad I_1 \leq 2 [\|b'\|_\infty^2 (t_2 - t_1)^2 + 4\alpha^2] \sup_{0 \leq s \leq t} \|DX_s\|_H^2.$$

Further observe

$$\begin{aligned} I_2 &\leq \left[\int_0^{t_1} (D_r X_{t_2} - D_r X_{t_1})^2 dr \right]^{\frac{1}{2}} \left[\int_0^{t_1} |D_r X_{t_2} + D_r X_{t_1}|^2 dr \right]^{\frac{1}{2}} \\ &\leq \sqrt{2} \left[\int_0^{t_1} (D_r X_{t_2} - D_r X_{t_1})^2 dr \right]^{\frac{1}{2}} \left[\int_0^{t_1} |D_r X_{t_2}|^2 + |D_r X_{t_1}|^2 dr \right]^{\frac{1}{2}} \\ &\leq \sqrt{2} \left[\int_0^{t_1} (D_r X_{t_2} - D_r X_{t_1})^2 dr \right]^{\frac{1}{2}} \left[\int_0^{t_2} |D_r X_{t_2}|^2 dr + \int_0^{t_1} |D_r X_{t_1}|^2 dr \right]^{\frac{1}{2}} \\ &\leq 2 \left[\int_0^{t_1} (D_r X_{t_2} - D_r X_{t_1})^2 dr \right]^{\frac{1}{2}} \sup_{0 \leq s \leq t} \|DX_s\|_H \\ &\leq 2 \left[\int_0^{t_2} (D_r X_{t_2} - D_r X_{t_1})^2 dr \right]^{\frac{1}{2}} \sup_{0 \leq s \leq t} \|DX_s\|_H, \end{aligned}$$

this inequality and (3.8) gives

$$I_2 \leq 2\sqrt{2[\|b'\|_\infty^2 (t_2 - t_1)^2 + 4\alpha^2]} \sup_{0 \leq s \leq t} \|DX_s\|_H^2.$$

Combining the estimates of I_1 and I_2 , we immediately get the desired inequality in the lemma.

It remains to prove (3.8). By (3.7), we have

$$\begin{aligned} (D_r X_{t_2} - D_r X_{t_1})^2 &\leq 2 \left| \int_{t_1}^{t_2} D_r b(X_s) ds \right|^2 + 2\alpha^2 \left| D_r \left(\sup_{0 \leq s \leq t_1} X_s \right) - D_r \left(\sup_{0 \leq s \leq t_2} X_s \right) \right|^2 \\ &\leq 2 \left| \int_{t_1}^{t_2} D_r b(X_s) ds \right|^2 + 4\alpha^2 \left| D_r \left(\sup_{0 \leq s \leq t_1} X_s \right) \right|^2 + 4\alpha^2 \left| D_r \left(\sup_{0 \leq s \leq t_2} X_s \right) \right|^2. \end{aligned}$$

By Hölder inequality, (3.2) and Proposition 1.1, we have

$$\begin{aligned}
\int_0^{t_2} \left| \int_{t_1}^{t_2} D_r b(X_s) ds \right|^2 dr &\leq \|b'\|_\infty^2 \int_0^{t_2} (t_2 - t_1) \int_{t_1}^{t_2} |D_r X_s|^2 ds dr \\
&= \|b'\|_\infty^2 (t_2 - t_1) \int_{t_1}^{t_2} \int_0^s |D_r X_s|^2 dr ds \\
&\leq \|b'\|_\infty^2 (t_2 - t_1)^2 \sup_{0 \leq s \leq t} \|DX_s\|_H^2.
\end{aligned}$$

Moreover, by (3.3) and (3.2) we have

$$\begin{aligned}
\int_0^{t_2} \left| D_r \left(\sup_{0 \leq s \leq t_2} X_s \right) \right|^2 dr &\leq \sup_{0 \leq s \leq t_2} \|DX_s\|_H^2 \leq \sup_{0 \leq s \leq t} \|DX_s\|_H^2, \\
\int_0^{t_2} \left| D_r \left(\sup_{0 \leq s \leq t_1} X_s \right) \right|^2 dr &= \int_0^{t_1} \left| D_r \left(\sup_{0 \leq s \leq t_1} X_s \right) \right|^2 dr \leq \sup_{0 \leq s \leq t} \|DX_s\|_H^2.
\end{aligned}$$

Collecting the above four inequalities, we immediately get the desired (3.8). \square

Lemma 3.3. *Let b be bounded smooth with $\|b'\|_\infty < \infty$, we have*

$$(3.10) \quad \sup_{0 \leq s \leq t} \|DX_s\|_H^2 \geq \frac{\sigma^2 t}{2(1 + 2\|b'\|_\infty^2 t^2 + 2\alpha^2)}, \quad t > 0.$$

Proof. By (3.7) and using $(a + b)^2 \geq \frac{1}{2}a^2 - b^2$, we have

$$\begin{aligned}
(D_r X_t)^2 &\geq \frac{1}{2}\sigma^2 - \left[\int_r^t D_r b(X_s) ds + \alpha D_r \left(\sup_{0 \leq s \leq t} X_s \right) \right]^2 \\
&\geq \frac{1}{2}\sigma^2 - 2 \left(\int_r^t D_r b(X_s) ds \right)^2 - 2\alpha^2 \left[D_r \left(\sup_{0 \leq s \leq t} X_s \right) \right]^2.
\end{aligned}$$

Further observe

$$\begin{aligned}
\int_0^t \left(\int_r^t D_r b(X_s) ds \right)^2 dr &\leq \int_0^t (t - r) \int_r^t |D_r b(X_s)|^2 ds dr \\
&\leq \int_0^t (t - r) \|b'\|_\infty^2 \int_r^t |D_r X_s|^2 ds dr \\
(3.11) \quad &\leq t \|b'\|_\infty^2 \int_0^t \int_r^t |D_r X_s|^2 ds dr \\
&= t \|b'\|_\infty^2 \int_0^t \|DX_s\|_H^2 ds \\
&\leq t^2 \|b'\|_\infty^2 \sup_{0 \leq s \leq t} \|DX_s\|_H^2,
\end{aligned}$$

where the second inequality is by Proposition 1.1. Hence,

$$\begin{aligned} \|DX_t\|_H^2 &\geq \frac{\sigma^2 t}{2} - 2\|b'\|_\infty^2 t^2 \sup_{0 \leq s \leq t} \|DX_s\|_H^2 - 2\alpha^2 \|D(\sup_{0 \leq s \leq t} X_s)\|_H^2 \\ &\geq \frac{\sigma^2 t}{2} - 2\|b'\|_\infty^2 t^2 \sup_{0 \leq s \leq t} \|DX_s\|_H^2 - 2\alpha^2 \sup_{0 \leq s \leq t} \|DX_s\|_H^2, \end{aligned}$$

where the last inequality is by (3.3).

This clearly implies

$$\sup_{0 \leq s \leq t} \|DX_s\|_H^2 \geq \frac{\sigma^2 t}{2} - 2\|b'\|_\infty^2 t^2 \sup_{0 \leq s \leq t} \|DX_s\|_H^2 - 2\alpha^2 \sup_{0 \leq s \leq t} \|DX_s\|_H^2,$$

which immediately yields the desired bound. \square

Lemma 3.4. *Let b be bounded smooth with $\|b'\|_\infty < \infty$ and $\sigma(x) \equiv \sigma$ with $\sigma \neq 0$. If $\alpha < 1$, $t_0 > 0$ and b satisfy*

$$\theta(t_0, \alpha, b) < 1/2$$

with $\theta(r, \alpha, b) := \sqrt{2\|b'\|_\infty^2 r^2 + 8\alpha^2} + \|b'\|_\infty^2 r^2 + 4\alpha^2$, then

$$(3.12) \quad \|DX_t\|_H^2 \geq \frac{[1 - 2\theta(t_0, \alpha, b)]\sigma^2 t}{2(1 + 2\|b'\|_\infty^2 t^2 + 2\alpha^2)}, \quad t \in [0, t_0].$$

Proof. Let $t \in [0, t_0]$. For all $0 \leq t_1 \leq t_2 \leq t$, by Lemma 3.2, we have

$$|\|DX_{t_2}\|_H^2 - \|DX_{t_1}\|_H^2| \leq 2\theta(t_2 - t_1, \alpha, b) \sup_{0 \leq s \leq t} \|DX_s\|_H^2.$$

Hence, for all $s \in [0, t]$,

$$\begin{aligned} \|DX_s\|_H^2 &\leq |\|DX_s\|_H^2 - \|DX_t\|_H^2| + \|DX_t\|_H^2 \\ &\leq 2\theta(t - s, \alpha, b) \sup_{0 \leq s \leq t} \|DX_s\|_H^2 + \|DX_t\|_H^2, \end{aligned}$$

and consequently

$$\sup_{0 \leq s \leq t} \|DX_s\|_H^2 \leq 2\theta(t - s, \alpha, b) \sup_{0 \leq s \leq t} \|DX_s\|_H^2 + \|DX_t\|_H^2.$$

The above inequality and (3.10) further give

$$\begin{aligned} \|DX_t\|_H^2 &\geq [1 - 2\theta(t - s, \alpha, b)] \sup_{0 \leq s \leq t} \|DX_s\|_H^2 \\ &\geq [1 - 2\theta(t_0, \alpha, b)] \sup_{0 \leq s \leq t} \|DX_s\|_H^2. \end{aligned}$$

Combining the above inequality and Lemma 3.3 immediately gives the desired inequality. \square

3.3. Multiplicative noise case. By the condition of σ , we have $\sup_{x \in \mathbb{R}} \sigma(x) < \infty$ or $\inf_{x \in \mathbb{R}} \sigma(x) > 0$. Without loss of generality, we assume that

$$\inf_{x \in \mathbb{R}} \sigma(x) > 0.$$

Let us consider the following well known transform

$$(3.13) \quad F(X_t) = \int_x^{X_t} \frac{1}{\sigma(u)} du,$$

it is easy to see that F is a strictly increasing function with bounded derivative. Hence,

$$(3.14) \quad \sup_{0 \leq s \leq t} F(X_s) = F\left(\sup_{0 \leq s \leq t} X_s\right).$$

By Itô formula, we have

$$(3.15) \quad F(X_t) = \int_0^t \left(\frac{b(X_s)}{\sigma(X_s)} - \frac{1}{2} \sigma'(X_s) \right) ds + B_t + \alpha \int_0^t \frac{1}{\sigma(X_s)} dM_s$$

where $M_t = \sup_{0 \leq s \leq t} X_s$. It is easy to see that M_t is an increasing function of t and that $\frac{1}{\sigma(X_s)}$ has a contribution to the related integral only when $X_s = M_s$. Hence,

$$(3.16) \quad F(X_t) = \int_0^t \left(\frac{b(X_s)}{\sigma(X_s)} - \frac{1}{2} \sigma'(X_s) \right) ds + B_t + \alpha \int_0^t \frac{1}{\sigma(M_s)} dM_s.$$

Since M_t is a continuous increasing function with respect to t , we have

$$(3.17) \quad F(X_t) = \int_0^t \left(\frac{b(X_s)}{\sigma(X_s)} - \frac{1}{2} \sigma'(X_s) \right) ds + B_t + \alpha \int_0^{M_t} \frac{1}{\sigma(u)} du.$$

By (3.14),

$$(3.18) \quad F(X_t) = \int_0^t \left(\frac{b(X_s)}{\sigma(X_s)} - \frac{1}{2} \sigma'(X_s) \right) ds + B_t + \alpha \sup_{0 \leq s \leq t} F(X_s).$$

Denote $Y_t = F(X_t)$, it solves the following perturbed SDE:

$$(3.19) \quad Y_t = \int_0^t \tilde{b}(Y_s) ds + B_t + \alpha \sup_{0 \leq s \leq t} Y_s$$

where $\tilde{b}(x) = \frac{b(F^{-1}(x))}{\sigma(F^{-1}(x))} - \frac{1}{2} \sigma'(F^{-1}(x))$. Applying Lemma 3.4, we get the following lemma about the dynamics Y_t :

Lemma 3.5. *Assume that b is bounded smooth and that σ is bounded smooth with $\|\sigma'\|_\infty < \infty$, $\|\sigma''\|_\infty < \infty$ and $\inf_{x \geq 0} |\sigma(x)| > 0$. Then \tilde{b} is bounded smooth. If $\alpha < 1$, $t_0 > 0$ and b satisfy*

$$\theta(t_0, \alpha, \tilde{b}) < 1/2$$

with $\theta(r, \alpha, \tilde{b}) := \left[\sqrt{2\|\tilde{b}'\|_\infty^2 r^2 + 8\alpha^2} + \|\tilde{b}'\|_\infty^2 r^2 + 4\alpha^2 \right]$, then

$$(3.20) \quad \|DY_t\|_H^2 \geq \frac{[1 - 2\theta(t_0, \alpha, \tilde{b})]t}{2(1 + 2\|\tilde{b}'\|_\infty^2 t^2 + 2\alpha^2)}, \quad t \in [0, t_0].$$

Proof. It is easy to check that under the conditions in the lemma \tilde{b} is bounded smooth with $\|\tilde{b}'\|_\infty < \infty$. Hence, the lemma immediately follows from applying Lemma 3.4 to Y_t . \square

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